The Finite Element Method for Fluid-Structure Interaction with open source software

Arbitrary Lagrangian Eulerian (ALE) methods

Riccardo Rossi Pooyan Dadvand

Kratos Team

Centre Internacional de Mètodes Numèrics en Enginyeria (CIMNE)

September 16, 2014, TU Munich

Outline

1. Introduction
2. Basic concepts
   - Basic definitions
3. Derivation
4. practical application

some symbols:

- The symbol $\mathbf{X}$ denotes the initial position of a material point, at the beginning of the time frame we consider
- The symbol $\mathbf{x}$ denotes a spatial point, that is, a point in the space. For example $\mathbf{x}(\mathbf{X}, t)$ will denote the position at the time step t of a particle that was originally in the position $\mathbf{X}$ of the space.
- The symbol χ denote the spatial position assumed at time t by a point whose position is fixed with respect to the mesh. For example if we consider that a point identified by χ coincides with a node of the mesh, the function $\mathbf{x}(\chi, t)$ tells us what is the position in space occupied by such node at the given time t. Since in principle the movement of the mesh *can be independent on the physical movement* at every time t such position referes *to a different material point*

(we follow here the presentation given in the book of Donea and Huerta)

Defining some operators

Let's define the operator ϕ which takes a material point and provides the corresponding spatial coordinate:

$$\phi : (\mathbf{X}, t) \longmapsto \phi(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) \quad (1)$$

this operator clearly defines a mapping between the undeformed space and the deformed one such that $\mathbf{x}(\mathbf{X}, t)$

In a similar way we can also define the operator $\mathbf{\Phi}$ defined such that

$$\mathbf{\Phi} : (\chi, t) \longmapsto \mathbf{\Phi}(\chi, t) = \mathbf{x}(\chi, t) \quad (2)$$

which describes the movement of the points that correspond to given mesh positions.

Defining some operators

if we assume that the mesh deformation is such that the mesh will not get inverted during the time span under consideration (which is a necessary condition to be able to perform calculations), it is also possible to define the operator $\boldsymbol{\Psi}^{-1}$ as

$$\boldsymbol{\Psi}^{-1} : (\mathbf{X}, t) \longmapsto \Psi^{-1}(\mathbf{X}, t) = \chi(\mathbf{X}, t) \tag{3}$$

which tells for any given material point $(\mathbf{X}, t)$ which is the position within the discretization.

Defining the Jacobians

For all of the mappings defined, we can define the jacobians.

$$\frac{\partial \phi}{\partial (\mathbf{X}, t)} = \frac{\partial (\mathbf{x}, t)}{\partial (\mathbf{X}, t)} = \begin{pmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{X}}|_t & \frac{\partial \mathbf{x}}{\partial t}|_{\mathbf{X}} \\ \frac{\partial t}{\partial \mathbf{X}}|_t & \frac{\partial \mathbf{t}}{\partial X}|_{\mathbf{X}} \end{pmatrix} \tag{4}$$

we can now recognize that the term $\frac{\partial \mathbf{x}}{\partial t}|_{\mathbf{X}}$ is the "material velocity" $\mathbf{v}$ which we associate to a given material particle originally located at the position $\mathbf{X}$. Substituting and simplifying we get

$$\frac{\partial \phi}{\partial (\mathbf{X}, t)} = \begin{pmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{X}}|_t & \mathbf{v} \\ \mathbf{0} & 1 \end{pmatrix} \tag{5}$$

Defining the Jacobians

similarly for the mapping Φ we can get

$$\frac{\partial \Phi}{\partial (\chi t)} = \frac{\partial (\mathbf{x}, t)}{\partial (\chi, t)} = \begin{pmatrix} \frac{\partial \mathbf{x}}{\partial \chi}|_t & \frac{\partial \mathbf{x}}{\partial t}|_\chi \\ \frac{\partial t}{\partial \chi}|_t & \frac{\partial \mathbf{t}}{\partial t}|_\chi \end{pmatrix} \tag{6}$$

We now introduce the symbol $\mathbf{v}_M$ to identify the "mesh velocity" as $\frac{\partial \mathbf{x}}{\partial t}|_\chi$. Taking this into account and simplifying, we get

$$\frac{\partial \Phi}{\partial (\chi, t)} = \begin{pmatrix} \frac{\partial \mathbf{x}}{\partial \chi}|_t & \mathbf{v}_M \\ \mathbf{0} & 1 \end{pmatrix} \tag{7}$$

Taking into account that in this case $\mathbf{x}$ shall be interpreted as $\mathbf{x}(\chi, t)$, the mesh velocity shall be interpreted as a measure of how fast the different positions of the mesh move in the space. This can be visualized considering the deformation of the solid attached to the mesh. The reader shall however take into account that this solid is moving INDEPENDENTLY on the material particles it describes.

Defining the Jacobians

We can also make the same exercise for the Ψ^{-1} operator to get

$$\frac{\partial \Psi^{-1}}{\partial (\mathbf{X}t)} = \frac{\partial (\chi, t)}{\partial (\mathbf{X}, t)} = \begin{pmatrix} \frac{\partial \chi}{\partial \mathbf{X}}|_t & \frac{\partial \chi}{\partial t}|_{\mathbf{X}} \\ \frac{\partial t}{\partial \mathbf{X}}|_t & \frac{\partial \mathbf{t}}{\partial t}|_{\mathbf{X}} \end{pmatrix} \tag{8}$$

If we introduce the symbol $\mathbf{w} = \frac{\partial \mathbf{X}}{\partial t}|_{\chi}$ and simplify, we get

$$\frac{\partial \Psi^{-1}}{\partial (\mathbf{X}t)} = \begin{pmatrix} \frac{\partial \chi}{\partial \mathbf{X}}|_t & \mathbf{w} \\ \mathbf{0} & 1 \end{pmatrix} \tag{9}$$

It is a little more involved to give a "physical" interpretation of the new variable **w**. Informally speaking this is a measure of how fast the mesh gets decoupled from the lagrangian movement.

Relation between the different velocities

One of the reasons to give the definitions in the last page, is that we can now seek for a relation between the various operators involved. The idea is that the operators described can be composed as $\phi = \mathbf{\Phi} \circ \mathbf{\Psi}^{-1}$. We can write this more explicitly as

$$\mathbf{x}\left(\mathbf{X}, t\right) = \mathbf{x}\left(\chi\left(\mathbf{X}, t\right), t\right) \tag{10}$$

or in terms of operators as:

$$\phi\left(\mathbf{X}, t\right) = \mathbf{\Phi}\left(\mathbf{\Psi}^{-1}\left(\chi, t\right), t\right) \tag{11}$$

by differentiating and using the chain rule we can write

$$\frac{\partial \phi}{\partial\left(\mathbf{X}, t\right)} = \frac{\partial \mathbf{\Phi}}{\partial\left(\chi, t\right)} \frac{\partial \mathbf{\Psi}^{-1}}{\partial\left(\mathbf{X}, t\right)} \tag{12}$$

Relation between the different velocities

If we now substitute in the chain rule the jacobians we wrote before, we can write the following matrix identity

$$\begin{pmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{X}}|_t & \mathbf{v} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathbf{x}}{\partial \chi}|_t & \mathbf{v}_M \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial \chi}{\partial \mathbf{X}}|_t & \mathbf{w} \\ \mathbf{0} & 1 \end{pmatrix} \tag{13}$$

doing the matrix multiplication on the right

$$\begin{pmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{X}}|_t & \mathbf{v} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathbf{x}}{\partial \chi}|_t \frac{\partial \chi}{\partial \mathbf{X}}|_t & \frac{\partial \mathbf{x}}{\partial \chi}|_t \mathbf{w} + \mathbf{v}_M \\ \mathbf{0} & 1 \end{pmatrix} \tag{14}$$

finally, comparing the entries at position 1-2 of the matrices, we can get the relation

$$\mathbf{v} - \mathbf{v}_M = \frac{\partial \mathbf{x}}{\partial \chi}|_t \mathbf{w} \tag{15}$$

ALE for arbitrary scalar functions

Let's now consider a given scalar function f (this could be for example the density distribution). Its values can be expressed as $f(\mathbf{X}, t)$ but also as $f(\mathbf{x}, t)$ or $f(\chi, t)$. To relate the different "points of view" on the same scalar, we can use differentiation

$$\frac{\partial f}{\partial(\mathbf{X}, t)} = \frac{\partial f}{\partial(\mathbf{x}, t)} \frac{\partial \phi}{\partial(\mathbf{X}, t)} \tag{16}$$

writing the same explicitly in matrix form, we get

$$\begin{pmatrix} \frac{\partial f}{\partial \mathbf{X}}|_t & \frac{\partial f}{\partial \mathbf{t}}|_{\mathbf{X}} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial \mathbf{x}}|_t & \frac{\partial f}{\partial \mathbf{t}}|_{\mathbf{x}} \end{pmatrix} \begin{pmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{X}}|_t & \mathbf{v} \\ \mathbf{0} & 1 \end{pmatrix} \tag{17}$$

doing the multiplication and looking again at the position 1-2

$$\frac{\partial f}{\partial \mathbf{t}}|_{\mathbf{X}} = \frac{\partial f}{\partial \mathbf{t}}|_{\chi} + \frac{\partial f}{\partial \mathbf{x}}|_t \cdot \mathbf{w} \tag{18}$$

which expresses the standard relation between the material time derivative $\frac{\partial \mathbf{f}}{\partial \mathbf{t}}|_{\mathbf{X}}$ (normally identified as $\frac{df}{dt}$) and the spatial one.

ALE for arbitrary scalar functions

The interesting fact is that a similar differentiation can be done as

$$\frac{\partial f}{\partial (\mathbf{X}, t)} = \frac{\partial f}{\partial (\chi, t)} \frac{\partial \mathbf{\Psi}^{-1}}{\partial (\mathbf{X}, t)} \tag{19}$$

writing the same explicitly in matrix form, we get

$$\begin{pmatrix} \frac{\partial f}{\partial \mathbf{X}}|_t & \frac{\partial f}{\partial \mathbf{t}}|_\mathbf{X} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial \chi}|_t & \frac{\partial f}{\partial \mathbf{t}}|_\chi \end{pmatrix} \begin{pmatrix} \frac{\partial \chi}{\partial \mathbf{X}}|_t & \mathbf{w} \\ \mathbf{0} & 1 \end{pmatrix} \tag{20}$$

doing explicitly the multiplication and looking and the entry at position 1-2 we then get

$$\frac{\partial f}{\partial \mathbf{t}}|_\mathbf{X} = \frac{\partial f}{\partial \mathbf{t}}|_\chi + \frac{\partial f}{\partial \chi}|_t \cdot \mathbf{w} \tag{21}$$

ALE for arbitrary scalar functions

this can be rewritten as

$$\frac{\partial f}{\partial \mathbf{t}}|_{\mathbf{X}} = \frac{\partial f}{\partial \mathbf{t}}|_{\chi} + \frac{\partial f}{\partial \mathbf{x}}|_{t} \frac{\partial \mathbf{x}}{\partial \chi}|_{t} \cdot \mathbf{w} \tag{22}$$

and finally, substituting 15, we obtain

the fundamental ALE equation

$$\frac{\partial f}{\partial \mathbf{t}}|_{\mathbf{X}} = \frac{\partial f}{\partial \mathbf{t}}|_{\chi} + \frac{\partial f}{\partial \mathbf{x}}|_{t} \cdot (\mathbf{v} - \mathbf{v}_M) \tag{23}$$

which is valid for any scalar function f, or for any component of a vector function. As a side note, this equation is typically written as

$$\frac{\partial f}{\partial \mathbf{t}}|_{\mathbf{X}} = \frac{\partial f}{\partial \mathbf{t}}|_{\chi} + (\mathbf{v} - \mathbf{v}_M) \cdot \nabla f \tag{24}$$

some physically relevant equations in ALE form

- mass conservation $\frac{\partial \rho}{\partial \mathbf{t}}|_\chi + \frac{\partial \rho}{\partial \mathbf{x}}|_t \left(\mathbf{v} - \mathbf{v}_M\right) \cdot \nabla \rho$
- linear momentum conservation $\frac{\partial \mathbf{v}}{\partial \mathbf{t}}|_\chi + \left(\mathbf{v} - \mathbf{v}_M\right) \cdot \nabla \mathbf{v} + \nabla \cdot \sigma = \mathbf{f}$
- convection-diffusion equation $\frac{\partial \phi}{\partial \mathbf{t}}|_\chi + \left(\mathbf{v} - \mathbf{v}_M\right) \cdot \nabla \phi + \nabla \cdot k \nabla \phi = \mathbf{q}$

how do we actually use such equations??

The usage of ALE equations is very simple: one shall prescribe a deformation of the mesh, for example by telling the position of the nodes of the mesh. The velocity of the mesh $\mathbf{v}_M$ shall be computed starting from such movement and *applying the same time integration scheme one would use for the lagrangian case*.

All gradients and areas needed to evaluate the discrete form shall be computed on the deformed mesh position, in a way completely equivalent to what would be done if the deformed mesh configuration was the eulerian domain of integration.

Lagrangian and Eulerian as particular cases

Lagrangian Case

We shall remark that the lagrangian case is recovered when the mesh is deformed such that $\mathbf{v}_M = \mathbf{v}$. It is trivial to show that in such case Eqn(23)reduces to

$$\frac{\partial \mathbf{f}}{\partial \mathbf{t}}|_{\mathbf{x}} = \frac{\partial \mathbf{f}}{\partial \mathbf{t}}|_{\chi} \tag{25}$$

proving the statement.

It is important to remark that, in order to recover the lagrangian case, the $\mathbf{v}_M$ shall be computed from the nodal displacements, so that an appropriate scheme, with its guaranteed order of accuracy (for example Newmark, Gear etc) is recovered in the fully lagrangian case. **the pratical implication is that the computation of the mesh velocity shall depend on the time integration scheme used in the fluid solver**.

We also remark that if this is not the case, prescribing a displacement on the boundaries may result in $\mathbf{v}_M \neq \mathbf{v}$ at the boundaries, implying that the behaviour is NOT lagrangian at the boundaries and that some sort of "leakage" may appear.

Lagrangian and Eulerian as particular cases

Eulerian Case

The eulerian limit is found when $\mathbf{v}_M = \mathbf{0}$. In such case Eqn(23)reduces to

$$\frac{\partial \mathbf{f}}{\partial \mathbf{t}}|_{\mathbf{x}} = \frac{\partial \mathbf{f}}{\partial \mathbf{t}}|_{\chi} + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}|_{t} \cdot (\mathbf{v}) \quad (26)$$

which is the standard eulerian form

Geometrical Conservation Law. Meaning & relevance

The "Geometrical Conservation Law" states that in order to recover at the very minimum first order accuracy, an ALE solver shall be able to **reproduce exactly a constant solution when subjected to an arbitrary deformation of the mesh** (provided of course that the mesh does not get inverted).

The relevance of this condition in practical considerations was discussed in the literature by many authors (particularly by Farhat), which concluded that for Finite Volume schemes, where equations are written in "conservative form" the requirement of the GCL was indeed crucial.

Within the FE community the relevance of the GCL was felt to be minor, with solver working reliably without having to do anything relevant to cope with the GCL. The root of this difference was found (see e.g. Forster) in the use of the non-conservative form of the equations, which makes easy to cope with the GCL at continuous level and renders the solvers less sensible to this aspect.

As a general statement, incompressible FE solvers based on the non-conservative form of the equations generally do not have problems with the GCL, on the contrary VOF solvers (based on the conservative form of the equations) generally suffer restrictions in dealing with GCL