The Finite Element Method for Fluid-Structure Interaction with open source software - Balance Equations and Their discretization

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Outline

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Balance Equations in conservation form

Let us start via a COMPLETELY GENERAL formulation for **any material**
Linear Momentum Conservation

$$\partial_t \rho \mathbf{u} + \nabla \left(\rho \mathbf{u} \otimes \mathbf{u} \right) + \nabla \cdot \sigma - \rho \mathbf{g} = \mathbf{0} \tag{1}$$

Mass Conservation

$$\partial_t \rho + \nabla \cdot \rho \mathbf{u} = 0 \tag{2}$$

Energy Conservation

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot \left(\mathbf{u} \left(\rho E + p \right) \right) = \rho \mathbf{g} \cdot \mathbf{u} + Q + \Psi + \nabla \cdot q \tag{3}$$

where $\mathbf{u}$ is the velocity, ρ is the density, p is the pressure, $\mathbf{g}$ is the gravity
The Energy is defined as $E := e + \rho \frac{\mathbf{u} \cdot \mathbf{u}}{2}$ (the sum of internal and kinetic energies), $\mathbf{q}$ is the heat flux vector, $\mathbf{Q}$ and Ψ the internal heat generation and internal heat dissipation functions respectively

Conservation form

The equations are written in the so-called "conservation form", which refers to an arrangement of an equation, that shows that a property represented is conserved by making its overall change equal to zero.

A state equation for the pressure and overall system

The problem needs to be completed by defining a relation between density, pressure and temperature.
We can generally assume that a relation of the type

$$\rho = \rho(p, T) \tag{4}$$

holds for a very wide class of fluids.
So we have 4 equations (momentum, mass, energy balance, state equations) and 4 unknowns (p, v, T, ρ). Note that it is possible to work with different sets of unknowns.

Coupled system

The system is completely coupled: one must solve all equations at once
It is impossible to solve one equation without solving the rest.

Assuming Newtonian Behaviour

A generic Cauchy stress tensor σ can be split into a deviatoric and an isochoric part as

$$\sigma = p\mathbf{I} + \tau \tag{5}$$

Newtonian fluids

are a particular class of materials for which it is possible to assume that

$$\tau = \mu \frac{\nabla \mathbf{u} + \nabla^T \mathbf{u}}{2} = \mu \nabla^s \mathbf{u} \tag{6}$$

μ is a measure of the resistance to shearing and takes the name of *dynamic viscosity*
For such class of fluids, the internal heat dissipation terms which appear into the energy equation is:

$$\psi = \nabla^s \mathbf{u} : \sigma \tag{7}$$

remark

The term $\nabla^s \mathbf{u}$ has a VERY STRICT relation to the elastic strain rate $\dot{\epsilon}$. Indeed it is exactly the same as the linear strain but *computed using velocities*

Incompressible flows

A large number of practical problems involve **incompressible flows**.

Incompressible flows

Flows where the density changes are negligible.

For example, water is usually modelled as incompressible or almost incompressible.
There is yet another way of looking at incompressible flows - based on the Mach number.

$$Ma = \frac{v}{c} \qquad (8)$$

where **u** and c are the velocity of the flow and the sound velocity in the considered medium.
When $Ma \approx 0$ the sound wave propagates in the flow instanteneously.

Compressible flows

Sub-sonic

When $Ma < 1$ one talks about the sub-sonic flows.
There information propagates faster than the moving object/flow.

The slow or still regions of the flow can adjust continuously to the arriving disturbance.

Super-sonic

When $Ma > 1$ one deals with supersonic flows.
Flow/object moves faster than sound. The medium in the upstream direction does not obtain the information upon its arrival prior to immediate contact,

So there are locations where the high speed and low speed coexist. At the boundary of the sub-sonic and super-sonic regions a shock occurs. Across a shock there is always an extremely rapid variation in pressure, temperature and density of the flow.

Momentun equation in the non-conservation form

Incompressible flows

Do not contain discontinuities. Are characterized by smooth solutions.

This allows us to rewrite the governing equations in the non-conservation form, which is convenient for the implementation. Let us see how this can be done.
We take the convective term $\nabla \cdot v \otimes v$ and perform partial integration. Possible due TO THE ASSUMPTION OF THE SMOOTHNESS of the variables and thus their differentiability:

$$\nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u})|_i = \partial_j (\rho \mathbf{u}_j \mathbf{u}_i) = \rho \mathbf{u}_j \partial_j \mathbf{u}_i + \mathbf{u}_i (\partial_j (\rho \mathbf{u}_j)) \tag{9}$$

Similarly we use the product rule for the inertia terms:

$$\partial_t (\rho \mathbf{u}_i) = \rho \partial_t \mathbf{u}_i + \mathbf{u}_i \partial_t \rho \tag{10}$$

Momentum equation in non-conservation form

The sum of the expanded convective and the inertia term give

$$\partial_t (\rho \mathbf{u}_i) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u})|_i = \rho \partial_t \mathbf{u}_i + \mathbf{u}_i \partial_t \rho + \rho \mathbf{u}_j \partial_j \mathbf{u}_i + \mathbf{u}_i \left(\partial_j \left(\rho \mathbf{u}_j\right)\right) = \\ = \rho \partial_t \mathbf{u}_i + \rho \mathbf{u}_j \partial_j \mathbf{u}_i + \mathbf{u}_i \left[\partial_t \rho + \partial_j \left(\rho \mathbf{u}_j\right)\right] \tag{11}$$

The term in parenthesis is nothing but the equation of mass conservation

$$\partial_t \rho + \partial_j \left(\rho \mathbf{u}_j\right) = 0 \tag{12}$$

and substituting it into Eq. (11) results in

$$\partial_t (\rho \mathbf{u}_i) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u})|_i = \rho \mathbf{u}_j \partial_j \mathbf{u}_i + \rho \partial_t \mathbf{u}_i \tag{13}$$

Momentum equation in non-conservation form

Adding the rest of the terms (which are equivalent in conservation and non-conservation forms) we obtain

$$\rho \mathbf{u}_j \partial_j \mathbf{u}_i + \rho \partial_t \mathbf{u}_i + \partial_i p - \partial_j \left(\mu \left(\partial_j \mathbf{u}_i + \partial_i \mathbf{u}_k \right) \right) = \rho \mathbf{b} \tag{14}$$

which is the non-conservation form of the momentum equation.

Non-conservation form of momentum balance

In symbolic notation it can be written as: $\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \nabla \mathbf{u} \cdot \mathbf{u} + \nabla p - \mu \nabla \cdot \left(\frac{\nabla \mathbf{u} + \nabla^T \mathbf{u}}{2} \right) = \rho \mathbf{b}$

Now we proceed with the simplification of the mass conservation equation

Mass conservation equation for incompressible and almost incompressible flows

The mass balance equation

$$\partial_t \rho + \nabla \cdot \rho \mathbf{u} = 0 \qquad (15)$$

can be simplified for the incompressible flow, as the incompressibity assumption means nothing but
$\partial_t \rho = 0$ and $\nabla \cdot \rho \mathbf{u} = \rho \nabla \cdot \mathbf{u}$
thus leading to the following form

Mass conservation for incompressible flows

$$\nabla \cdot \mathbf{u} = 0 \qquad (16)$$

Navier-Stokes equations

Looking at the momentum and continuity equations for the incompressible Newtonian fluid one can see that they contain only two variables: $\mathbf{u}$ and p

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\left(\mathbf{u}\cdot\nabla\right)\mathbf{u} + \nabla p - \mu\nabla\cdot\left(\frac{\nabla\mathbf{u} + \nabla^T\mathbf{u}}{2}\right) = \rho\mathbf{b} \tag{17}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{18}$$

$$\tag{19}$$

This means that if the flow is isothermal, these two equations are necessary and sufficient for describing the mechanical system. The energy equation becomes "uncoupled" from the system.
Thus for the incompressible flow, the 4 governing equations (momentum, energy, mass balance and state equation) reduced into two (momentum and mass).
This is known as incompressible Navier-Stokes equations. It is a non-linear (note the convective term) coupled dybamic system.
NB N-S eqs are thoroghly studied and much better understood from both the physical and the mathematical point of view than the general compressible problem.

Remark: incompressible equations in the presence of thermal effects

If the flow is exposed to heat, but the encountered temperature gradients are small, one often uses the so called **Boussinesq approximation**
The governing equations remain incompressible, but the gravity term “adjusts” to account for thermal effects.

Boussinesq approximation

$$\mathbf{g}^{new} = \mathbf{g}_s(1 - \alpha(T_{new} - T_s)) \tag{20}$$

α is the Boussinesq coefficient and index “s” stands for standard conditions. In all the rest of the terms, the density changes are neglected. Thus the Navier-Stokes equations under Boussinesq assumption read

Navier-Stokes+Boussinesq

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho\left(\mathbf{u} \cdot \nabla\right)\mathbf{u} + \nabla p - \mu \nabla \cdot \left(\frac{\nabla \mathbf{u} + \nabla^T \mathbf{u}}{2}\right) = \rho \mathbf{b}_s(1 - \alpha(T_{new} - T_s))$$
$$\nabla \cdot \mathbf{u} = 0 \tag{21}$$

Discretization of incompressible Navier-Stokes equations

Let us proceed with discretization...

The strong statement of the boundary value problem

Let us start by formalizing the mathematical description of the problem of interest. Navier-Stokes equations define a boundary value problem, which can be stated as:

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\left(\mathbf{u}\cdot\nabla\right)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{b}$$
$$\mathbf{u}\cdot\nabla\rho = 0 \qquad (22)$$

N

ote that we used the following approximation
$\mu\nabla\cdot\left(\frac{\nabla\mathbf{u}+\nabla^T\mathbf{u}}{2}\right) \approx \mu\Delta\mathbf{u}$

To ensure that our system has a unique solution, and to make the problem well posed it is necessary to prescribe exactly one boundary condition at each part of the boundary.

Boundary conditions

$\mathbf{u} = \mathbf{u}_D$ on Γ_D(Dirichlet boundary conditions for velocities) $n\cdot\sigma = t_n$ on Γ_N(Neumann boundary conditions for normal tractions)

Note that $n\cdot\sigma = -pn + 2\nu n\cdot\nabla^S\mathbf{u}$

Weak statement of the problem

Weak form can be obtained as follows: we multiply the momenum and the continuity equations by the velocity and pressure test functions: ω and q. The task is to:

Weak form

Find v and p such that

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t},\omega\right)+\rho\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}-\mu\left(\Delta\mathbf{u},\omega\right)+\left(\nabla p,\omega\right)=\rho\langle\mathbf{f},\omega\rangle \tag{23}$$

$$\left(q,\nabla\cdot\mathbf{u}\right) \tag{24}$$

The brackets stand for the integrals, e.g.: $(\nabla p,\omega)=\int_\Omega \nabla p\cdot\omega d\Omega$

Weak form

Next we perform the partial integration of the viscous term

Integration by parts: viscous term

$$\mu\left(\Delta\mathbf{u},\omega\right) = -\mu\left(\nabla\mathbf{u},\nabla\omega\right) + \mu\int_\Gamma \omega \boldsymbol{n}\cdot\nabla\mathbf{u}$$

where Γ is the boundary of the domain Ω

Boundary terms

- On the Dirichlet boundary: $\int_{\Gamma_D}\omega \boldsymbol{n}\cdot\nabla\mathbf{u} = 0$
- The Neumann part remains: $\mu\int_\Gamma \omega \boldsymbol{n}\cdot\nabla\mathbf{u}$

Weak form

And now we perform the partial integration of the pressure gradient term
$(\nabla p, \omega) = -(p, \nabla \omega) + \int_\Gamma \omega n \cdot p d\Omega$
Thus the we obtain the following weak-form of the problem

Final weak form of Navier-Stokes equations

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t}, \omega \right) - \mu \left(\nabla \mathbf{u}, \nabla \omega \right) - (p, \nabla \omega) = \rho \langle \mathbf{f}, \omega \rangle + (\omega, t)_{\Gamma_N} \tag{25}$$

$$(q, \nabla \cdot \mathbf{u}) = 0 \tag{26}$$

From weak form to space-discretized Navier-Stokes equations

The Galerkin formulation of Navier-Stokes equations leads to a mixed finite element method.
We introduce approximations for the velocity $\mathbf{u}_h$ and the pressure p_h and the associated weight function ω_h and q_h. The spaces these functions are taken from (let us call them $\mathcal{V}^h$ and Q^h) are the finite dimensional subspaces of $\mathcal{V}$ and Q
The Galerkin form of Navier-Stokes equations is thus written as

Galerkin form of Navier-Stokes equations

$$\left(\rho \frac{\partial \mathbf{u}_h}{\partial t}, \omega_h\right) - \mu\left(\nabla \mathbf{u}_h, \nabla \omega_h\right) - \left(p_h, \nabla \omega_h\right) = \rho\langle f, \omega_h\rangle + \left(\omega_h, t\right)_{\Gamma_N} \quad (27)$$

$$\left(q_h, \nabla \cdot \mathbf{u}_h\right) = 0 \quad (28)$$

Note that the equations are still continuous in time

Space-discretized Navier-Stokes equations

Next step consists in approximating the velocity and the pressrue in terms of shape functions and associated nodal values.

FE interpolations

$p(x) = \sum_I N_I(x) \cdot p_I$ and $v(x) = \sum_I N_I(x) \cdot v_I$
$q(x) = \sum_I N_I(x) \cdot q_I$ and $\omega(x) = \sum_I N_I(x) \cdot \omega_I$

N are the standard linear shape functions and I is the nodal index. We use Einstein's summation convention.
We susbstitute our interpolations into the Galerkin form and keeping in mind that the equations must hold for any q and ω.

Space-discretized Navier-Stokes equations: matrix form

Plugging in the shape functions into discrete weak form we obtain the following matrix form of the incompressible Navier-Stokes equations: (note that from now onwards we ommit the nodal indices, but remember that $\mathbf{u}$ and p stand for $\mathbf{u}_h^I$ and p_h^I)

Matrix format of Navier-Stokes equations

$$\rho \mathbf{M} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{C}(\mathbf{u})\mathbf{u} - \mu \mathbf{L}\mathbf{u} + \mathbf{G}p = \mathbf{f} \quad (29)$$

$$\mathbf{D}\mathbf{u} = 0 \quad (30)$$

Space-discretized Navier-Stokes equations: matrix form

Matrix format of Navier-Stokes equations

$$\rho \mathbf{M} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{C}(\mathbf{u})\mathbf{u} + \mu \mathbf{L}\mathbf{u} + \mathbf{G}p = \mathbf{f} \tag{31}$$

$$\mathbf{D}\mathbf{u} = 0 \tag{32}$$

where

$$\mathbf{M} = \rho \int_{\Omega_e} \mathbf{N}\mathbf{N}^T d\Omega \tag{33}$$

$$\mathbf{L} = \int_{\Omega_e} \nabla\mathbf{N}\nabla\mathbf{N}^T d\Omega \tag{34}$$

$$\mathbf{G} = -\int_{\Omega_e} \nabla\mathbf{N}\mathbf{N} d\Omega \tag{35}$$

$$\mathbf{C}(\mathbf{u}) = \rho \int_{\Omega_e} \mathbf{N}\left(\mathbf{u} \cdot \nabla\mathbf{N}\right) d\Omega \tag{36}$$

$$\mathbf{F} = \int_{\Omega_e} \mathbf{N}\mathbf{f} d\Omega + \int_{\Gamma_N} n \cdot (p - \mu \nabla \mathbf{u}) \tag{37}$$

$$\mathbf{D} = -\mathbf{G}^T \tag{38}$$

Remark: on the viscous term

We remind that we used the following approximation:

approximation

$$\mu\nabla\cdot\left(\frac{\nabla\mathbf{u}+\nabla^T\mathbf{u}}{2}\right)\approx\mu\Delta\mathbf{u}\to\mu\mathbf{L}$$

thus using the so-called "Laplacian" form of the viscous term
This is commonly done in CFD codes.
Nevertheless we could use the "exact" or "symmetric gradient" form, giving

Symmetric gradient form of viscous term

$$\mu\nabla\cdot\left(\frac{\nabla\mathbf{u}+\nabla^T\mathbf{u}}{2}\right)\to\mathbf{A}$$

with

Symmetric gradient form of viscous term

$$\mathbf{A}=\int_\Omega B^T\mathbf{C}_\mu B d\Omega$$

where B is the strain velocity matrix (identical to the strain-displacement matrix of elasticity) and $\mathbf{C}_\mu$ is the viscous constitutive matrix.

Problems of Galerkin FE formulation of Navier-Stokes equations

It turns out that Galerkin formulation of Navier-Stokes equations leads to two types of instabilities:

Problems of Galerkin FEM for Navier-Stokes

1. Oscillations in the velocity field in convection-dominated flows (similar to those in convection-diffusion equation)
2. Pressure wiggles for some velocity-pressure interpolations pairs

The origin of the first instability was discussed in the section devoted to the convection-diffusion equation.
Here we have a closer look at the second, pressure instability.
For the sake of clarity we illustrate it using a particular case of Navier-Stokes equations, namely the Stokes problem.

Stokes problem

When the Reynold's number of a flow is very low

Reynold's number

$Re = \frac{\rho \mathbf{u} D}{\mu}$

the nonlinear terms due to inertial effects can be neglected

- $\rho \left(\mathbf{u} \cdot \nabla \right) \mathbf{u} = 0$
- $\frac{\partial \mathbf{u}}{\partial t}$

resulting in a linear problem.

$$-\mu \Delta \mathbf{u} + \nabla p = \rho \mathbf{b} \quad (39)$$

$$\mathbf{u} \cdot \nabla \rho = 0 \quad (40)$$

$$(41)$$

Stokes problem: discrete form

The discrete version of Stokes problem can be written as:

Discrete Stokes problem

$$\mathbf{K}\mathbf{u} + \mathbf{G}p = \mathbf{f} \tag{42}$$

$$\mathbf{D}\mathbf{u} = 0 \tag{43}$$

where $\mathbf{K} = -\mu\mathbf{L}$ is the viscosity matrix, $\mathbf{G}$ and $\mathbf{D}$ are the gradient and divergence matrices and $\mathbf{f}$ is the force vector.

Stokes problem: solvability

We can write the Stokes problem in the following matrix form:

$$\begin{pmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{D} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ 0 \end{pmatrix} \tag{44}$$

The structure of the discrete system is peculiar due to the presence of null-matrix in the lower diagonal.

Solvability of Stokes problem

It can be shown that this algebraic system can only be solved provided that the **kernel** of the gradient matrix **G** is zero.

This would lead to the solvability of the global system and thus to the unique solution for the velocity and pressure.

Solvability of Stokes problem

To understand better what influence the kernel of the gradient operator has upon the solvability of the system, let us assume that the system is solved in a partitioned manner. This is done by obtaining from the first equation

$$\mathbf{u} = \mathbf{K}^{-1} (\mathbf{F} - \mathbf{G}p) \tag{45}$$

and plugging it into the second equation resulting in the so-called "pressure shur equation"

$$-\mathbf{D}\mathbf{K}^{-1}\mathbf{G}p = \mathbf{D}\mathbf{K}^{-1}\mathbf{F} \tag{46}$$

Condition for the solvability

The "pressure shur" equation can be solved if the matrix $\mathbf{D}\mathbf{K}^{-1}\mathbf{G}$ is positive definite which assures its **invertibility**. This is satisfied **if and only if** the discrete gradient $\mathbf{G}$ has zero kernel.

Solvability of Stokes problem

According to the mathematical theorem known as Ladyzhenskaya-Babushka-Brezzi (LBB) condition, the kernel of **G** depends on the choice of the interpolation spaces for velocity and the pressure.
The velocity-pressure interpolation pairs must taken from the velocity and pressure spaces ($\mathcal{V}^h$ and $\mathcal{Q}^h$) that respect the following inf-sup condition:

LBB velocity-pressure compatibility condition

$$inf_{q^h \in \mathcal{Q}^h} \, sup_{w^h \in \mathcal{V}^h} \frac{(q^h, \nabla \cdot w^h)}{||q||_0 ||w^h||_1} > \alpha > 0 \quad (47)$$

where α is a positive constant independent of the mesh size h. Only certain combinations of velocity-pressure interpolations satisfy the LBB compatibility condition.

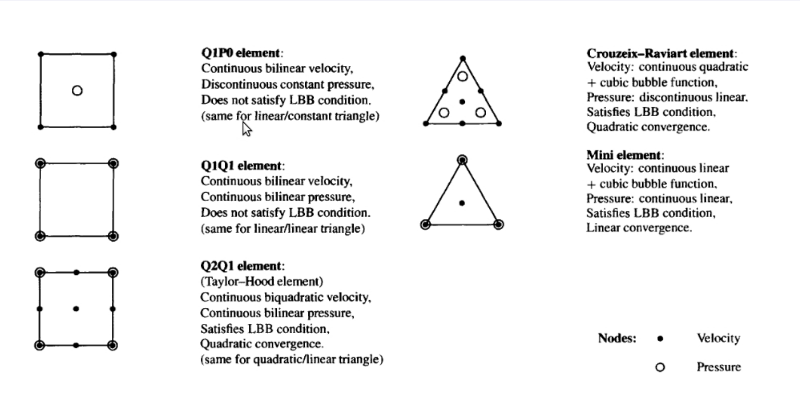
Stokes problem: solvability

Unfortunately the most widely used linear velocity-pressure interpolation pairs do not satisfy the LBB condition.
Using LBB stable elements, such as Q2Q1 (Taylor-Hood element, based upon continuous quadratic velocity, continuous bilinear pressure) or mini element (continuous linear velocity+bubble function, continuous linear pressure) is less favorable due to the complexity of the associated implementation and clearly lower computational efficiency than that of simplicial elements.
As it is well known, there exist possibilities for circumventing the LBB condition, thus permitting the use of the velocity-pressure pairs that are unstable when standard Galerkin formulations are used.
The basic idea behind pressure stabilization methods is to modify the weak form of the problem in such a way that the positive definiteness of the matrix

$$\begin{pmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{D} & 0 \end{pmatrix}$$

Stokes problem: solvability

Some velocity-pressure pairs



Euler Problem

The Euler problem is the limit reached by the NS equations as $\nu \to 0$
Although stabilized methods will work for arbitrary boundary conditions on velocity, the "native" condition is the slip BC

Boundary Conditions

The Navier Stokes problem needs to be completed by appropriate boundary conditions.

1. velocities need to be imposed on the "Dirichlet Boundary". Note that velocities should go towards the inside of the domain and NOT towards the outside
2. tractions should be imposed on the neumann boundary. The problem is ONLY defined if velocities point outwards at the neumann boundary (remedies are possible for this).
3. it is possible to combine the two conditions via the so called Robin boundary condition –¿ not the subject of this course

remark

Boundary conditions need to be imposed on ALL the boundary (althought for example the zero traction condition may be automatically imposed)

is structure different from fluid?

Apart for the convective term and the eulerian disctretization? no!!
If the Fluid is written in a lagrangian framework it is NOTHING ELSE than a structure with a **simple** constitutive law

Creating Stable elements

In order to use equal order velocity-pressure pairs, we need to "stabilize" the formulation. This can be achieved by modifying consistently the continuum form of the equations, in a way that is somewhat similar to what was done to resolve the convection diffusion problem (althought in this case the purpose is different)

Problem Statement

- Navier-Stokes equation consists in finding $\mathbf{u}$ and p:

$$\partial_t \mathbf{u} - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in}\Omega, \qquad t \in]0, T[\tag{48}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in}\Omega, \qquad t \in]0, T[\tag{49}$$

$$\mathbf{u} = \mathbf{u_0} \qquad \text{in}\Omega, \qquad t = 0 \tag{50}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{in}\Gamma, \qquad t \in]0, T[\tag{51}$$

$\mathbf{u}$ and p: Velocity and pressure (unknowns)
$\mathbf{f}$ and $\mathbf{u}_0$: Body forces and initial conditions (data)
ν : Viscosity (physical parameter)

Variational Form

- The *variational* equivalent of the **NS** is:

find **u** and p such that for all $\{\mathbf{v}, q\} \in H_0^1 \times L^2$

$$\int_\Omega v\,\partial_t u + \int_\Omega \nu \nabla u \nabla v + \int_\Omega v u \cdot \nabla u - \int_\Omega p \nabla \cdot u = \int_\Omega f\, v \tag{52}$$

$$\int_\Omega q \nabla \cdot u = 0 \tag{53}$$

- let us define the following notation:

$$\mathcal{L}(\mathbf{U}) := \begin{bmatrix} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p \\ \nabla \cdot \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} := \mathbf{F} \tag{54}$$

in which $U = [\mathbf{u}, p]$. Defining $\mathbf{V} = [\mathbf{v}, q]$

Bilinear Form

- Using predefined notation

$$\mathbf{B}(\mathbf{U},\mathbf{V}) := \int_\Omega \mathbf{v}\,\partial_t \mathbf{u} + \int_\Omega \nu \nabla \mathbf{u}\,\nabla \mathbf{v} + \int_\Omega \mathbf{v}\mathbf{u}\cdot\nabla \mathbf{u} - \int_\Omega \mathbf{p}\,\nabla\cdot\mathbf{u} + \int_\Omega \mathbf{q}\nabla\cdot\mathbf{u} \quad (55)$$

and the linear form $L(V) := \int_\Omega f, v$. Problem (54) with the homogeneous Dirichlet condition consists then in finding $U \in \mathcal{W}_0$ such that

$$B(\mathbf{U},\mathbf{V}) = L(\mathbf{V}) \quad (56)$$

The bilinear form could now be solved numerically by approximating velocity and pressure domains using standard Galerkin method. Discrete version of problem (41) is to find appropriate $U_h \in \mathcal{W}_{h,0}$ such that

$$B(\mathbf{U}_h,\mathbf{V}_h) = L(\mathbf{V}_h) \quad (57)$$

Standard Galerkin method ⇒ Instabilities, locking
Galerkin method for cavity problem:

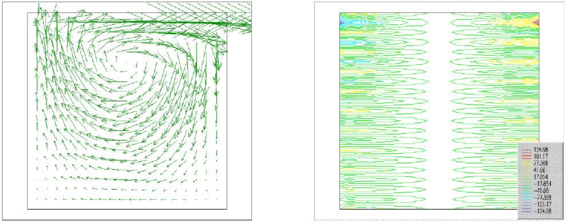


Figure : Spurious pressure

Stabilization

- **Algebraic Sub Grid Scale** *(ASGS)*

The solution is divided to two parts:

1. One that is captured by FEM
2. The subscale part

- Decomposition :

$$\mathcal{W} = \mathcal{W}_h \oplus \tilde{\mathcal{W}}$$

$\mathcal{W}_h \Rightarrow$ the space covered by finite element discretization
$\tilde{\mathcal{W}} \Rightarrow$ that completes $\mathcal{W}_h$ in $\mathcal{W}$ $\Rightarrow$ *sub grid scales*

$B(\mathbf{U}, \mathbf{V}) = L(\mathbf{V})$ is then replaced by:

$$B(\mathbf{U}_h, \mathbf{V}_h) + B(\tilde{\mathbf{U}}, \mathbf{V}_h) = L(\mathbf{V}_h) \qquad \forall \mathbf{V}_h \in \mathcal{W}_{h,0}, \tag{58}$$

$$B(\mathbf{U}_h, \tilde{\mathbf{V}}) + B(\tilde{\mathbf{U}}, \tilde{\mathbf{V}}) = L(\tilde{\mathbf{V}}) \qquad \forall \tilde{\mathbf{V}} \in \tilde{\mathcal{W}}_0. \tag{59}$$

Stabilization cont.

- The idea now is to compute $\tilde{\mathbf{U}}$ from the second equation and replace it to the first one

Integration by part:

$$B(\mathbf{U}_h, \mathbf{V}_h) + \sum_K \int_K \tilde{\mathbf{U}} \cdot \mathcal{L}^*(\mathbf{V}_h)\, d\Omega + \sum_K \int_{\partial K} \tilde{\mathbf{u}} \cdot (q_h \mathbf{n} + \mu \mathbf{n} \cdot \nabla \mathbf{v}_h)\, d\Gamma = L(V_h) \quad (60)$$

$$\sum_K \int_{\partial K} \tilde{\mathbf{v}} \cdot (p\mathbf{n} + \mu \mathbf{n} \cdot \nabla \mathbf{u})\, d\Gamma + \sum_K \int_K \tilde{\mathbf{V}} \cdot \mathcal{L}(\tilde{\mathbf{U}})\, d\Omega = \sum_K \int_K \tilde{\mathbf{V}} \cdot (\mathbf{F} - \mathcal{L}(\mathbf{U}_h))\, d\Omega \quad (61)$$

- summations over all elements, n is unit normal

$\mathcal{L}^*$ is the formal adjoint of $\mathcal{L}$, given by

$$\mathcal{L}^*(\mathbf{V}_h) = \begin{bmatrix} -\nu \Delta \mathbf{v}_h - \mathbf{a} \cdot \nabla \mathbf{v}_h - \nabla q_h \\ -\nabla \cdot \mathbf{v}_h \end{bmatrix} \quad (62)$$

- Calculate **sub scales**, $\tilde{U}$, from equation(61)

$\Downarrow$

Plug them into equation (60).

Stabilization cont.

- Considering a continuous flux through element boundaries:

$$\sum_K \int_K \tilde{\mathbf{V}} \cdot \mathcal{L}(\tilde{\mathbf{U}})\, d\Omega = \sum_K \int_K \tilde{\mathbf{V}} \cdot (\mathbf{F} - \mathcal{L}(\mathbf{U}_h))\, d\Omega \tag{63}$$

sub scales equation is equivalent to satisfy in **every element**,

$$\mathcal{L}(\tilde{\mathbf{U}}) = \mathbf{R} := \mathbf{F} - \mathcal{L}(\mathbf{U}_h) \qquad \text{in} \Omega_K \tag{64}$$

- Once solved this equation for $\tilde{\mathbf{U}}$, the only thing that remained is to plugging it in to equation (60). One possible approximation for $\tilde{\mathbf{U}}$ in every element is

$$\tilde{\mathbf{U}} \approx \tau_K \mathbf{R} \qquad \text{in} K \tag{65}$$

- Many discussions on the choice of diagonal matrix τ_K,

$$\tau_K = \begin{pmatrix} \tau_{1,t} & 0 \\ 0 & \tau_2 \end{pmatrix}$$

$$\tau_{1,t} = \left(\frac{1}{\theta \delta t} + \frac{1}{\tau_1}\right)^{-1} \qquad \tau_1 = \left[\frac{4\nu}{h^2} + \frac{2|\mathbf{a}|}{h}\right]^{-1} \tag{66}$$

$$\tau_2 = \nu + \frac{h|\mathbf{a}|}{2} \tag{67}$$

Stabilization cont.

- Using approximation(65) for the *sub scales* and neglecting the boundary integrals

$$B(\mathbf{U}_h, \mathbf{V}_h) + \underbrace{\sum_K \int_K \tau_K (\mathbf{F} - \mathcal{L}(\mathbf{U}_h)) \cdot \mathcal{L}^*(\mathbf{V}_h)\, d\Omega}_{ASGS\ Stabilization} = L(V_h) \quad (68)$$

The expanded form of stabilized NS equation is:
Let's define: $\mathbf{Res} := \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + \mathbf{a} \cdot \nabla \mathbf{u} + \nabla p - \mathbf{f}$

$$\int_\Omega \mathbf{v}\, \partial_t \mathbf{u} + \int_\Omega \nu \nabla \mathbf{u}\, \nabla \mathbf{v} + \int_\Omega \mathbf{v}\, \mathbf{u} \cdot \nabla \mathbf{u} - \int_\Omega p\, \nabla \cdot \mathbf{u} + \int_\Omega q \nabla \cdot \mathbf{u} \quad (69)$$

$$+ \int_\Omega (\nu \Delta \mathbf{v} + \mathbf{a} \cdot \nabla \mathbf{v} + \nabla q) \cdot \tau_1 \mathbf{Res}$$

$$+ \int_\Omega \nabla \cdot \mathbf{v}\, (\tau 2 \nabla \cdot \mathbf{u}) = \int_\Omega \mathbf{f}\, \mathbf{v} \quad (70)$$

Time Scheme

- Semi-discrete approach $\Rightarrow$ ODE system of nonlinear equations

$$\mathbf{M}\dot{\mathbf{U}} + \mathbf{N}(\mathbf{U}) + \mathbf{K}(\mathbf{Y}) = \mathbf{F} \tag{71}$$

- Bossak scheme α-modified version of newmark family.

$$(1-\alpha_\beta)\mathbf{M}\dot{\mathbf{U}}_{n+1} + \alpha_\beta \mathbf{M}\dot{\mathbf{U}}_n + \mathbf{N}(\mathbf{U}_{n+1}) + \mathbf{K}(\mathbf{Y}_{n+1}) = \mathbf{F}_{n+1} \tag{72}$$

$$\dot{\mathbf{U}}_{n+1} = \frac{1}{\gamma \Delta t}(\mathbf{U}_{n+1} - \mathbf{U}_n) - \frac{1-\gamma}{\gamma}\dot{\mathbf{U}}_n \tag{73}$$

$$\mathbf{Y}_{n+1} = \mathbf{Y}_n + \Delta t \mathbf{U}_n + \Delta t^2 [\frac{1-2\beta}{2}\dot{\mathbf{U}}_n + \beta \dot{\mathbf{U}}_{n+1}] \tag{74}$$

α_β algorithmic parameter.

- Unconditional stability, second-order accuracy and maximal high-frequency

$$\alpha_\beta \leq 0, \qquad \gamma = \frac{1}{2} - \alpha_\beta, \qquad \beta = \frac{1}{4}(1-\alpha_\beta) \tag{75}$$

Time Scheme

- To treat **nonlinearities** $\Rightarrow$ A predictor-multi-corrector
- Prediction phase,

$$\mathbf{U}_{n+1}^{(0)} = \mathbf{U}_n \quad (76)$$

A consistent prediction of acceleration and displacement

$$\dot{\mathbf{U}}_{n+1}^{(0)} = \frac{\gamma - 1}{\gamma}\dot{\mathbf{U}}_n \qquad \mathbf{Y}_{n+1}^{(0)} = \mathbf{Y}_n + \Delta t \mathbf{U}_n + \frac{1}{2}\Delta t^2 (1 - \frac{2\beta}{\gamma}) \quad (77)$$

And the residual at iteration "i"

$$\mathbf{G}^{(i)}(\dot{\mathbf{U}}_{n+\alpha_\beta}^{(i)}, \mathbf{U}_{n+1}^{(i)}, \mathbf{Y}_{n+1}^{(i)}) = \mathbf{M}\dot{\mathbf{U}}_{n+\alpha_\beta}^{(i)} + \mathbf{N}(\mathbf{U}_{n+1}^{(i)}) + \mathbf{K}(\mathbf{Y}_{n+1}^{(i)}) - \mathbf{F}_{\mathbf{n+1}}^{(\mathbf{i})} \quad (78)$$

Linearization

- A Newton's linearizion of $\mathbf{G}^{(i)}$ with respect to velocity,

$$\mathbf{G}^{(i)}(\dot{\mathbf{U}}^{(i)}_{n+\alpha_\beta}, \mathbf{U}^{(i)}_{n+1}, \mathbf{Y}^{(i)}_{n+1}) + \frac{\partial \mathbf{G}^{(i)}(\dot{\mathbf{U}}^{(i)}_{n+\alpha_\beta}, \mathbf{U}^{(i)}_{n+1}, \mathbf{Y}^{(i)}_{n+1})}{\partial \mathbf{U}^{(i)}_{n+1}} \Delta \mathbf{U}^{(i)}_{n+1} = \mathbf{0} \tag{79}$$

Non-linear residual + tangent matrix ∗ increment of the solution

$$\Downarrow$$

A linear system for each corrector step

$$\mathbf{A}^{(i)} \Delta \mathbf{U}^{(i)}_{n+1} = -\mathbf{G}^{(i)}, \quad \mathbf{A}^{(i)} = \frac{1}{\gamma \Delta t}\mathbf{M} + \mathbf{N}^{(i)}_{n+1} + \frac{\Delta t \beta}{\gamma}\mathbf{K}^{(i)}_{n+1} \tag{80}$$

Once (79) is solved the solution is updated,

$$\mathbf{U}^{(i+1)}_{n+1} = \mathbf{U}^{(i)}_{n+1} + \Delta \mathbf{U}^{(i)}_{n+1} \tag{81}$$

Matrix Form

$$\mathbf{A}^{(i)} = \frac{1}{\gamma \Delta t}\mathbf{M} + \mathbf{N}_{n+1}^{(i)} + \frac{\Delta t \beta}{\gamma}\mathbf{K}_{n+1}^{(i)} \tag{82}$$

$$\mathbf{M} = \begin{pmatrix} \mathbf{M_u} + \tau_1 \mathbf{S}_{adv} & \vdots & \mathbf{0} \\ \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots \\ \tau_1 \mathbf{S}_{grad} & \vdots & \mathbf{0} \end{pmatrix}, \quad \mathbf{K} = \mathbf{0} \tag{83}$$

$$\mathbf{M_u} = \int_\Omega \rho \mathbf{N}_u \mathbf{N}_v \, d\Omega$$

$$\mathbf{S}_{adv} = \int_\Omega \mathbf{a} \cdot \nabla \mathbf{N}_v \mathbf{N}_u, \, d\Omega \qquad \mathbf{S}_{grad} = \int_\Omega \nabla \mathbf{N}_q \mathbf{N}_u, \, d\Omega$$

Matrix Form Cont.

$$\mathbf{A}^{(i)} = \frac{1}{\gamma \Delta t}\mathbf{M} + \mathbf{N}_{n+1}^{(i)} + \frac{\Delta t \beta}{\gamma}\mathbf{K}_{n+1}^{(i)} \tag{84}$$

$$\mathbf{N} = \left(\begin{array}{c|c} \mathbf{C}_{\mu} + \mathbf{C}_{adv} + \tau_2 \mathbf{S}_{Div} & -\mathbf{D} + \tau_1 \mathbf{S}_{adv}^{grad} \\ \hline \mathbf{D}^T + \tau_1 \mathbf{S}_{grad}^{avd} & \tau_1 \mathbf{L} \end{array} \right) \tag{85}$$

$$\mathbf{C}_{\mu} = \int_{\Omega} \mu \nabla \mathbf{u} \nabla \mathbf{v} \qquad \mathbf{C}_{adv} = \int_{\Omega} \mathbf{N}_v \mathbf{a} \cdot \nabla \mathbf{N}_u$$

$$\mathbf{D} = \int_{\Omega} \mathbf{N}_p, \nabla \cdot \mathbf{N}_v \, d\Omega \qquad \mathbf{L} = \int_{\Omega} \nabla \mathbf{N}_q, \nabla \mathbf{N}_p, d\Omega \qquad \mathbf{G} = \int_{\Omega} \mathbf{N}_u, \nabla \mathbf{N}_q, d\Omega$$

$$\mathbf{S}_{Div} = \int_{\Omega} \nabla \cdot \mathbf{u} \, \nabla \cdot \mathbf{v}, d\Omega \qquad \mathbf{S}_{grad}^{avd} = \int_{\Omega} \nabla \mathbf{N}_q \mathbf{a} \cdot \nabla \mathbf{N}_u \qquad \mathbf{S}_{avd}^{grad} = (\mathbf{S}_{grad}^{avd})^T$$