The Finite Element Method for Fluid-Structure Interaction with open source software - The Convection-Diffusion Problem

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Convection-diffusion equation

The unsteady Convection-diffusion equation is:

$$\begin{aligned} \partial_t u + \mathbf{a} \cdot \nabla u - \nabla \cdot \kappa \nabla u + \sigma u &= q \text{ in } \Omega,\ t \in (0, T); \\ u(\mathbf{x}, 0) &= 0 \quad \text{ in } \Omega, \\ u &= u_D \quad \text{ in } \Gamma_D, \\ \kappa \mathbf{n} \cdot \nabla u &= h \quad \text{ in } \Gamma_N \end{aligned} \tag{1}$$

1. $\mathbf{a}$ is the velocity field, κ is the viscous term and σ the reaction term.
2. Initial solution at the entire domain Ω in addition to the Dirichlet, Γ_D, and Neumann, Γ_N, BC's are considered.

The equivalent *weak* form of this set of equation is:
Find $u(\mathbf{x}, t) \in \mathcal{S}(t)$ such that for all $w \in \mathcal{V}$ we have:

$$\int_\Omega w\partial_t u + \int_\Omega w\mathbf{a} \cdot \nabla u + \int_\Omega \nabla w \kappa \nabla u + \int_\Omega w\sigma u = \int_\Omega wq + \int_{\Gamma_N} wh \qquad (2)$$

• Various time discretization techniques can be used but as we are more interested on the spatial discretization and related stabilization terms, a *smooth* solution in time is considered and therefore we have:

$$\mathcal{S}(t) = \{u \,|\, u(.,t) \;\in\; H \,,\; u(\mathbf{x},t) = u_D \, on \, \mathbf{x} \in \Gamma_D\}$$

• Space of solution, $\mathcal{S}(t)$, is defined as all *smooth*, $u \in H$, that satisfies the Dirichlet condition.
• Space of test functions, $\mathcal{V}$, then is defined as:

$$\mathcal{V} = \{w \,|\, w \;\in\; H \,,\; w(\mathbf{x}) = 0 \, on \, \mathbf{x} \in \Gamma_D\}$$

- Spatial discretization of the convection-diffusion equation (2) by means of Galerkin finite elements method exhibits instabilities.

$$\Downarrow$$

- Oscillations appear as the convective effect dominant the viscous effects.

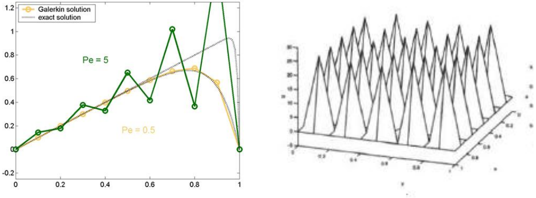


Figure : Instabilities in 1D and 2D cases

- To better understand this phenomena Let's consider the 1D *steady* case:

$$\begin{array}{c} au_x - \kappa u_{xx} = q, \ \text{ in }]0, L[; \\ u(x,0) = 0 \ \text{ in } \Omega \\ u = 0.0 \ \text{ in } x = 0\,,\, x = L \end{array} \tag{3}$$

The weak form then is:

$$\int_0^L w a u_x \, dx + \int_0^L w_x \kappa u_x \, dx = \int_0^L w q \, dx \tag{4}$$

Spatial discretization is performed considering the uniform linear element of size h

• Shape functions in the normalized coordinate ξ for node 1 and node 2 of each element have the form

$$N_1(\xi) = \frac{1}{2}(1-\xi)\ ,\ \ N_2(\xi) = \frac{1}{2}(1+\xi) \quad -1 \leq \xi \leq +1$$

The *weak form* (4) for an arbitrary element takes the following form:

$$\int_l^{l+h} N_s\, a\, \frac{dN_t}{dx}\, dx + \int_l^{l+h} \frac{dN_s}{dx}\, \kappa\, \frac{dN_t}{dx}\, dx = \int_l^{l+h} N_s q\, dx \qquad (5)$$

$$s, t = 1, 2$$

• The unknown u and the position x at each elements are discretized using the same shape functions as :

$$u(x) = N_1\, u_1 + N_2\, u_2 \quad x = N_1\, x_1 + N_2\, x_2$$

Therefore we have for the derivatives:

$$dx = \frac{h}{2} d\xi \quad \frac{dN_s}{dx} = \frac{dN_s l}{d\xi}\frac{d\xi}{dx} = \frac{2}{h}\frac{dN_s}{d\xi}$$

• The elemental convection, C^e, and diffusion, K^e, matrices for each element can be computed from(5)

$$C^e = \frac{a}{2}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \quad K^e = \frac{\kappa}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{6}$$

Assembling for each element the final stencil for node j is:

$$a\frac{u_{j+1}-u_{j-1}}{2h} - \kappa\left(\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}\right) = \frac{1}{6}(q_{j-1}+4q_j+q_{j+1}) \tag{7}$$

• More insight on the effect of the convection and diffusion on the Galerkin solution is reached by introducing the Péclet number:

$$P_e = \frac{ah}{2\kappa}$$

- This number expresses the ratio of convective to diffusive transport. This allows us to rewrite the stencil(7) in the form:

$$\frac{a}{2h}\left(\frac{P_e - 1}{P_e} u_{j+1} + \frac{2}{P_e} u_j - \frac{P_e + 1}{P_e} u_{j-1}\right) = \frac{1}{6}(q_{j-1} + 4q_j + q_{j+1}) \tag{8}$$

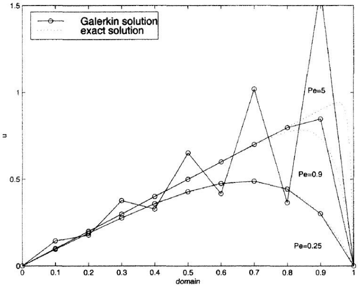
- Consider our 1D example with $q = 1$ and $L = 1$

- A uniform source has been chosen in this manner, the *truncation error*, which will arise from the Galerkin discretization of our model problem, is only attributed fully to the discrete representation of the *convection* and *diffusionoperators*.

- The exact solution to this model problem is given by:

$$u(x) = \frac{1}{a}\left(x - \frac{1-e^{\gamma x}}{1-e^{\gamma}}\right) \quad \gamma = a/\kappa \tag{9}$$

- The numerical approximation obtained from (8) for several values of Péclet number is:



- Galerkin solution is corrupted by non-physical oscillations when the Péclet number is larger than one.

Non-symmetric convection operator dominates the diffusion operator

- **Why? What does Galerkin stencil lack to recover the stable solution?**
- Knowing the analytic solution, *exact* stencil can be obtained for our 1D example :

$$\frac{a}{2h}\left([1-coth(P_e)]u_{j+1}+[2coth(P_e)]u_j-[1+coth(P_e)]u_{j-1}\right)=1 \qquad (10)$$

This stencil is similar to the Galerkin one. We can rearrange this equation to see the similarities and differences with the Galerkin discretization as following:

$$a\frac{u_{j+1}-u_{j-1}}{2h}-(\kappa+\bar{\kappa})(\frac{u_{j+1}-2u_j+u_{j-1}}{h^2})=1.0 \qquad (11)$$

$$\bar{\kappa}=\beta\frac{ah}{2} \quad \beta=coth(P_e)-\frac{1}{P_e}$$

Lack of $\bar{\kappa}$ in Galerkin discretization produces oscillation especially as the P_e increases.

Multidimensional case

- The extension to multidimensional domains of the concept of modified weighting functions discussed above is not trivial

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The balancing diffusion should be added
in the flow direction only, and not transversely.

$$\Downarrow$$

Construct the artificial diffusion operator in tensorial form

$$\Downarrow$$

$$\nu_{ij} = \bar{\nu} a_i a_j / \|\mathbf{a}\|^2$$

Adding a viscosity of this form to the steady weak form we have:

$$\int_\Omega w\mathbf{a}\cdot\nabla u + \int_\Omega \nabla w(\kappa I + \bar{\nu})\nabla u = 0 \tag{12}$$

If we focus on the new term we observe that:

$$\int_\Omega \nabla w \bar{\nu} \nabla u d\Omega = \int_\Omega \frac{\bar{\nu}}{||\mathbf{a}||^2}(\mathbf{a}\cdot\nabla w)(\mathbf{a}\cdot\nabla u) \tag{13}$$

Therefore we can rewrite equation (12) as:

$$\int_\Omega (w + \underbrace{\frac{\bar{\nu}}{||\mathbf{a}||^2}\mathbf{a}\cdot\nabla w}_{\textit{Streamline Upwind}})\mathbf{a}\cdot\nabla u + \int_\Omega \kappa\nabla w\cdot\nabla u = 0 \tag{14}$$

We can see that the stabilization only affect the convective term.

- Galerkin and Streamline Upwind (SU) shape functions for the 1D case:

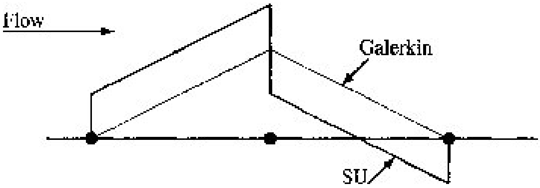

Fig. 2.11 Weighting function of the streamline-upwind (SU) method for linear elements.

- Different stabilization techniques for 1D convection-diffusion-reaction problem

$$\mathbf{a} \cdot \nabla u - \nabla \cdot (\nu \nabla u) + \sigma u = s \quad \text{in } \Omega,$$

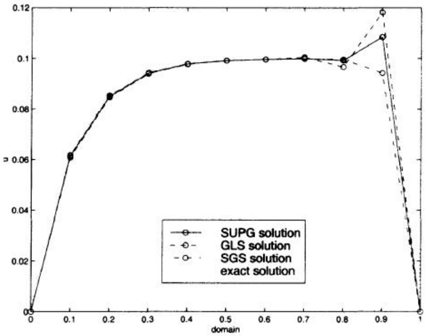

Fig. 2.14 Comparison between SUPG (solid line), GLS (dashed line) and SGS (dash–dot line) solutions of the convection–diffusion–reaction problem (2.54) for $a = 1$, $\nu = 10^{-2}$, $\sigma = 10$ and a uniform mesh of 10 linear elements. The dotted line shows the exact solution.

Stabilization

- We see that stabilization is provided by adding sufficient diffusion in the flow direction.
- *Consistent stabilization* is obtained by using the full residual of the differential equation.

Let's define the residual of the differential equation (1), $\mathcal{R}(u)$, as:

$$\mathcal{R}(u) = \partial_t u + \mathbf{a} \cdot \nabla u - \nabla \cdot \kappa \nabla u + \sigma u - q$$

- Note that this residual can be calculated inside each element.
- *Consistent stabilization* techniques takes the following form:

$$\int_\Omega w \partial_t u + \int_\Omega w \mathbf{a} \cdot \nabla u + \int_\Omega \nabla w \kappa \nabla u + \int_\Omega w \sigma u + \underbrace{\sum_e \int_{\Omega_e} \mathcal{P}(w) \tau \mathcal{R}(u) d\Omega}_{\textit{Stabilization terms}} = \int_\Omega wq + \int_{\Gamma_N} wh$$

- Different *consistent stabilization* techniques can be obtained base on the definition of operator $\mathcal{P}(w)$

- The SUPG method: $\mathcal{P}(w) = \mathbf{a} \cdot \nabla w$
So the stabilization term take the form:

$$\sum_e \int_{\Omega_e} \mathcal{P}(w) \tau \mathcal{R}(u) d\Omega =$$
$$\sum_e \int_{\Omega_e} \mathbf{a} \cdot \nabla w \, \tau \left[\partial_t u + \mathbf{a} \cdot \nabla u - \nabla \cdot \kappa \nabla u + \sigma u - q \right] \quad (15)$$

  – In case that linear element are used the second order term $\nabla \cdot \kappa \nabla u$ disappears.

- The Galerkin/Least-squares (GLS) method:

$$\mathcal{P}(w) = \mathbf{a} \cdot \nabla w - \nabla \cdot \kappa \nabla w + \sigma w$$

– Stabilization parameter τ is defined as:

$$\tau = \left(\frac{1}{\delta t} + \frac{2\|\mathbf{a}\|}{h} + \frac{4\kappa}{h^2} + \sigma\right)^{-1}$$

Here δt is a measure of the time step and h represents the element size.

- 2D model example:

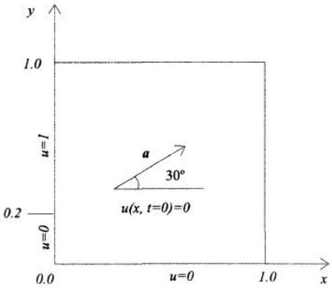

Fig. 2.18 Convection of discontinuous inlet data skew to the mesh: problem statement.

- Different stabilization techniques:

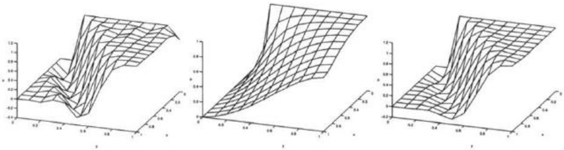

Fig. 2.19 Galerkin (left), artificial diffusion (center) and SUPG (right) solutions for the 2D convection–diffusion problem with downwind natural conditions.

Galerkin method $\Rightarrow$ *spurious oscillations*

Artificial diffusion method $\Rightarrow$ *over diffusive*

SUPG method $\Rightarrow$ *Sufficient diffusion*