# The Finite Element Method for Fluid-Structure Interaction with open source software - Basics

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# Outline

# Eulerian vs Lagrangian kinematical descriptions

Two main "viewpoints" are commonly used in the literature:

1. **Lagrangian viewpoint** which consists in following the material particles in all of their motion
2. **Eulerian viewpoint** which focuses on a fixed portion of space and attempts to determine the physical quantitiens associated to the fluid particles which pass at any given time on the top of the control region

as we shall comment later there exist a third viewpoint, called Arbitrary Lagrangian Eulerian viewpoint which is similar in spirit to the eulerian but allows taking into account a motion of the reference domain.

# Mathematical Preliminaries

A list of the necessary results is given below. .

- **Divergence Theorem**

$$\int_{\Omega} \frac{\partial g_i(\mathbf{x})}{\partial x_i} d\Omega = \int_{\Gamma} n_i g_i d\Gamma \quad or \quad \int_{\Omega} \nabla \cdot g(\mathbf{x}) d\Omega = \int_{\Gamma} n \cdot g d\Gamma \tag{1}$$

- **Gauss Theorem**

$$\int_{\Omega} \frac{\partial g_i(\mathbf{x})}{\partial x_j} d\Omega = \int_{\Gamma} n_j g_i d\Gamma \quad or \quad \int_{\Omega} \nabla g(\mathbf{x}) d\Omega = \int_{\Gamma} n \otimes g d\Gamma \tag{2}$$

# Mathematical Preliminaries

- **Green-Gauss theorem**

$$-\int_\Omega \omega \nabla^2 \mathbf{u} d\Omega = -\int_\Omega \nabla\omega \cdot \nabla u - \nabla \cdot \omega \nabla u d\Omega$$
$$= \int_\Omega \nabla\omega \cdot \nabla u d\Omega - \int_\Gamma \omega \left( n \cdot \nabla u \right) d\Gamma \tag{3}$$

- **Derivative of the determinant of the Jacobian**

$$j := det(\mathbf{F}) \rightarrow \frac{Dj}{Dt} \equiv \dot{j} = j \nabla \cdot v \tag{4}$$

- **Reynolds transport theorem**

$$\frac{D}{Dt} \int_{\Omega(t)} f(\mathbf{x}, t) d\Omega = \int_{\Omega(t)} \frac{\partial f}{\partial t} d\Omega + \int_\Gamma (\mathbf{v} \cdot \mathbf{n}) f dS = \int_{\Omega(t)} \frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{v} f) \tag{5}$$

# Material and spatial derivatives

in the lagrangian frame we normally express the time derivative as $\frac{D\phi}{Dt}$ where $\phi = \phi(\mathbf{x}, t)$ can be taken as a function of the time and of the current position of the particle of interest. When working in the Eulerian frame we will still focus on a position $\mathbf{x}$ of the space, but we will take into account that different particles pass through this point at different instants in time.
in any case if we take the function $\phi$ and we differentiate it in time we obtain that

$$\frac{\partial\phi(\mathbf{x}, t)}{\partial t} = \frac{\partial\phi(\mathbf{x}, t)}{\partial t}|_x + \frac{\partial\phi(\mathbf{x}, t)}{\partial\mathbf{x}}\frac{\partial\mathbf{x}}{\partial t}|_t \quad (6)$$

since the term $\dfrac{\partial\mathbf{x}}{\partial t}$ represents the lagrangian velocity $\mathbf{v}$ of the particle that occupies the position $x$ at time $t$
we can substitute this definition to obtain

$$\frac{D\phi(\mathbf{x}, t)}{Dt} = \frac{\partial\phi(\mathbf{x}, t)}{\partial t}|_x + \mathbf{v}\cdot\frac{\partial\phi(\mathbf{x}, t)}{\partial\mathbf{x}} \quad (7)$$

or syntethically

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \mathbf{v}\cdot\nabla\phi \quad (8)$$

this second form is known as the **spatial time derivative** and is normally used in the eulerian framework

# Conservation Equations in Eulerian Form - Mass Conservation

The mass contained in any given domain can be expressed as

$$m(\Omega) = \int_\Omega \rho(X,t)\, d\Omega(t) \tag{9}$$

where $\rho(\mathbf{X},t)$ is the density of the material point $\mathbf{X}$. The conservation implies that the mass is conserved on any given domain, in symbols

$$0 = \frac{Dm(\Omega(t))}{Dt} = \frac{D}{Dt}\int_\Omega \rho(X,t)\, d\Omega = \int_\Omega \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) d\Omega \tag{10}$$

Where we applied the Reynolds transport theorem. It is also readily verified that for continuous $\rho$, taking into account that $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho$ the former can be written as

$$\int_\Omega \frac{D\rho}{Dt} + \rho \nabla \cdot v d\Omega \tag{11}$$

the arbitrariety of the domain ensures that the relation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot v = \frac{D\rho}{Dt} + \rho \nabla \cdot v = 0 \tag{12}$$

holds pointwise (note that the equalities above are only verified fro continuous density functions).

# Conservation Equations in Eulerian Form - Conservation of Linear Momentum

the linear momentum is by definition

$$\mathbf{p}_{lin}(\Omega, t) := \int_\Omega \rho \mathbf{v} d\Omega \tag{13}$$

by introducing the resultant of the forces acting on the given control volume

$$\mathbf{f}(t) := \int_\Omega \rho b d\Omega + \int_\Gamma \mathbf{t} d\Gamma \tag{14}$$

The second Newton's law of motion (the momentum conservation for a continuum) takes the form

$$\frac{D\mathbf{p}_{lin}(\Omega, t)}{Dt} = \mathbf{f}(t) \tag{15}$$

differentiating the 13 and using the Reynolds transport theorem yields

$$\frac{D\mathbf{p}_{lin}(\Omega, t)}{Dt} = \int_\Omega \frac{D\rho\mathbf{v}}{Dt} + \rho\mathbf{v}\nabla \cdot v d\Omega = \int_\Omega \rho\frac{D\mathbf{v}}{Dt} + \mathbf{v}\left(\frac{D\rho}{Dt} + \rho\nabla \cdot v\right) d\Omega \tag{16}$$

# Conservation Equations in Eulerian Form - Conservation of Linear Momentum

Taking into account that the conservation of mass requires $\frac{D\rho}{Dt} + \rho\nabla \cdot v = 0$ we obtain

$$\frac{D\mathbf{p}_{lin}(\Omega, t)}{Dt} = \int_{\Omega} \rho \frac{\partial \mathbf{v}}{\partial t} d\Omega \quad (17)$$

on the other hand using the divergence theorem

$$\int_{\Gamma} \mathbf{t} d\Gamma = \int_{\Gamma} n \cdot \sigma d\Gamma = \int_{\Omega} \nabla \cdot \sigma d\Omega \quad (18)$$

grouping together and considering the arbitrariety of the domain we obtain finally

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \sigma + \rho \mathbf{b} \quad (19)$$

which is the strong form for the conservation of linear momentum